Fermion families, and chirality through extra dimensions

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Abstract

We give a simple model to explain the origin of fermion families, and chirality through the use of a domain wall placed in a five dimensional space-time.

1 Introduction

The popularity of higher dimensional spaces has revived in the last years by the hope that the extra space dimensions other than the the usual three dimensions may be experimentally accessible in near future. In this context the idea of the confinement of the usual particle spectrum into a four dimensional topological defect of the higher dimensional space-time is also revived especially in the view that by using Randall-Sundrum [1] type spaces one can confine gravity as well in

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infinite dimensions. In this study we shall consider a metric with a 4-dimensional Poincare invariance [2] and a domain wall in a Randall-Sundrum type space. We find that it gives some important clues towards the understanding of the origin of fermion families and chirality. By using a Minkowski-like metric we get a domain wall solution in five dimensions where fermions may be concentrated at more than one point of the wall. Each concentration point corresponds to a fermion family. After adding a fermion mass term to this scheme the left-handed and right-handed fermions become concentrated at different regions in the wall. In other words the wall itself acts as a mother 3-brane which carries two sub-branes, right-handed and left-handed. Each of these two sub-branes contain n different symmetrical sub-subbranes whose locations can be identified with different masses of different fermion families. Finally we include gravity into the picture by using a Randall-Sundrum-like metric. The framework employed here has some conceptual similarities with the study of Dvali and Shifman, which considers families as neighbors in a multi-brane world in five dimensions [3]. Another study with some similar aspects is by Arkani-Hamed and Schmaltz [4] where fermion mass hierarchies[1] are explained by the exponentially suppressed overlap of fermion wave functions located at different points in the extra dimension(s). We will give a comparison of these studies with the present one at the end of the next section after we give the essential points of this study.

2 Basic Lines of The Model

Consider the following well-known 5-dimensional scalar Lagrangian [6]

$$L = \frac{1}{2}\partial_A\phi\partial^A\phi - \frac{1}{4}\lambda(\phi^2 - \frac{\mu^2}{\lambda})^2 \ , \quad A = 0,1,2,3,4 \tag{1}$$

with

$$ds^2 = \eta_{AB}dx^A dx^B = dx_\mu \, dx^\mu - (3ay^2 + b^2)^2 dy^2 \ , \quad \mu = 0,1,2,3 \tag{2}$$

[1] In fact the idea of a better undertanding of fermion masses and mixings through higher dimensional spaces is not new. Among these, the studies in the context of heterotic string orbifolds seem to be especially appealing [5].

where a, b are some constants and $(\eta_{AB}) = diag(1, -1, -1, -1, -1)$ The equation of motion corresponding to Eq.(1), and its domain and anti-domain wall solutions [7] are

$$(\frac{1}{3ay^2+b})\frac{\partial}{\partial y}[(\frac{1}{3ay^2+b})\frac{\partial}{\partial y}\phi] + \mu^2\phi - \lambda\phi^3 = 0 \ , \tag{3}$$

$$\phi_{cl} = \pm\frac{\mu}{\sqrt{\lambda}}tanh\,[\frac{\mu}{\sqrt{2}}(ay^3+by+c)] \ ; \tag{4}$$

respectively.

We take the following fermion-scalar interaction Lagragian[2] .

$$\begin{aligned} & i\bar{\Psi}\Gamma^{\mu}\,D_{\mu}\Psi + i\bar{\Psi}\Gamma^{4}\,\frac{1}{(3ay^2+b)}\frac{\partial\Psi}{\partial y} + g\bar{\Psi}\phi\Psi \\ & \text{where}\ \ \Gamma^4 = -i\gamma_5 \ \ , \quad D_{\mu} = \partial_{\mu} + igB_{\mu} \end{aligned} \tag{5}$$

In the presence of the domain wall, the Dirac equation is

$$i\gamma^{\mu}\,D_{\mu}\Psi + \frac{1}{3ay^2+b}\gamma_5\frac{\partial\Psi}{\partial y} + g\phi_{cl}\Psi = 0 \tag{6}$$

Assume that

$$i\gamma^{\mu}\,D_{\mu}\Psi = m[\eta(y) - f]\gamma_5\Psi \tag{7}$$

where $\eta(y)$=$\frac{\mu}{\sqrt{2}}|(ay^3+by+c)|$ (the absolute value results from the orbifold symmetry introduced in the next section). This equation can be obtained, for example, by introducing an auxillary fermionic field, χ coupling to the other fields only through the following Lagrangian

$$L = \frac{1}{2}\partial_A\sigma\partial^A\sigma + i\bar{\chi}\not{D}\Psi + i\bar{\Psi}\not{D}\chi - \alpha\bar{\chi}\sigma\gamma_5\Psi - \alpha\bar{\Psi}\sigma\gamma_5\chi \tag{8}$$

where $\alpha = \frac{m\sqrt{\lambda}}{\mu}$ and $\not{D} = \gamma^{\mu}\,D_{\mu}$. (One should take $\alpha << 1$ in order to not conflict with phenomenlogy. This is plausible because $\frac{m\sqrt{\lambda}}{\mu}$ is a large number

[2]A similar Lagrangian is considered in [8]. In fact one can identify ϕ in this equation as the gauge field corresponding to a sixth dimension. This can be, for example, done by embedding this five dimensional space in a six dimensional space studied by Manton [9] where instead of taking the extra dimensions be compact one should only assume rotational symmetry. In that case $B_5 = \phi$ in Eq.(5) should be replaced by $\Phi + \tilde{\Phi}$ of Ref.[9]. However for the sake of simpilcity we take this term arise from a general scalar-fermion interaction term.

in general as is evident from Eq.(17) in the next section.) After using Eq.(7), Eq.(6) reduces to

$$\frac{\partial \Psi_L}{\partial y} - \frac{\partial \Psi_R}{\partial y} + (3ay^2+b)(-\alpha\sigma_{cl}+g\phi_{cl})\Psi_R + (3ay^2+b)(\alpha\sigma_{cl}+g\phi_{cl})\Psi_L = 0 \quad (9)$$

where $\Psi_L = \frac{1}{2}(1-\gamma_5)\Psi$, $\Psi_R = \frac{1}{2}(1+\gamma_5)\Psi$. The solutions of Eq.(9) are

$$\begin{aligned} & \Psi_R = exp[\beta\, ln(e^{\eta}+e^{-\eta})\psi_R \\ & = (e^{\eta}+e^{-\eta})^{\beta} e^{-m(\frac{1}{2}\eta^2 - f\eta + d)]}\psi_R \\ & \Psi_L = exp[-\beta\, ln(e^{\eta}+e^{-\eta}) - m(\frac{1}{2}\eta^2 - f\eta + d)]\psi_L \\ = \quad & (e^{\eta}+e^{-\eta})^{-\beta} e^{-m(\frac{1}{2}\eta^2 - f\eta + d)]}\psi_L \end{aligned} \quad (10)$$

where $\beta = g\frac{\mu}{\sqrt{\lambda}}$ and ψ is the solution of $i\gamma^{\mu} D_{\mu}\psi = m(\eta(y) - f)\gamma_5\psi$. The solutions are well behaving provided $m > 0$. One of $(e^{\eta}+e^{-\eta})^{(\pm\beta)}$ is a monotonically increasing function of η and the other is a monotonically decreasing function of η while $e^{-m(\frac{1}{2}\eta^2 - f\eta + d)]}$ is a monotonically decreasing function provided $f < 0$, $d > 0$. Therefore Ψ_L is a monotonically decreasing function of η (provided $\beta > 0$) with its only maximum at $\eta = z_L = 0$ while Ψ_R has only one maximum at $\eta = z_R > 0$. We notice that Ψ_L and Ψ_R are concentrated at different locations of the wall (that is at the maxima of Ψ_L and Ψ_R) and they have different distributions. Moreover there are different fermions corresponding to different roots of $\eta = z_{L(R)}$ with exactly the same probablity distrubutions at different locations of the wall. One can assume that al the roots of z_L (z_R) are much closer to each other than the roots of z_R (z_L) and the roots of z_L are much closer to zero than the roots of z_R. Then all the fermion families corresponding to the roots of z_L, that is Ψ_L's, will be almost at the center of the wall while Ψ_R's will be somewhat farther than the others. In other words the probability of Ψ_R participating in the interactions whose gauge bosons well localised in the wall will be signaficantly reduced. The number of the roots of z_L, z_R are three and can considered as locations of different fermion families. Another interesting aspect of the above equations is that they give mass-like terms for chiral fermions while, as long as we are aware, the previous solutions are given

for massless chiral fermions [8,10]. The importance of this construction is that one can embed this model in a six or higher dimensional model. Then these mass-like terms will contribute to mass matrix which will give the masses for the physical fermions after diagonalization. In the case of six dimensions such a scheme will result in the usual fermions and their mirrors with the same masses. So physically relavant models need to assign the fermions and their mirrors to different gauge groups. In the case of seven or higher dimensional models it is possible to give the fermions and their mirrors different masses provided the entries of the mass matrix are taken as general complex numbers. To be more precise let us consider the following seven dimensional Dirac equation

$$\Gamma^A \, D_A \Psi = 0 \tag{11}$$

$$\text{where} \quad \Gamma^A \, D_A = \begin{pmatrix} V & iD_5 + D_6 \\ iD_6 - D_6 & -V \end{pmatrix}$$

$$V = i\gamma_5 \, D_4 + \gamma^\mu \, D_\mu \quad , \quad D_A = \partial_A + igB_A \quad , \quad A = 0, 1, ..., 6 \ , \quad \mu = 0, 1, 2, 3$$

If either the gauge bosons corresponding to extra dimensions have vacuum expectation values or the derivatives give mass terms due to compactification of the extra dimensions or due to both this equation induces a mass matrix

$$M = \begin{pmatrix} i\gamma_5 m_4 & im_5 + m_6 \\ im_5 - m_6 & -i\gamma_5 m_4 \end{pmatrix} \tag{12}$$

This equation leads to two different fermions, usual fermions and their mirrors, with different masses provided you take m_i's as arbitrary complex numbers. Both masses become the same if one reduces the dimension of the space-time to six or let all m_i's be real. Another interesting feature of the model is that some of the fermions can be localized in the domain wall as some others are localized in the anti-domain wall, corresponding to the same scalar field because the mass term in the exponent dominates over ϕ_{cl} to insure the convergence of the argument of the exponent while this not possible for the massless solutions of Dirac equation because in that case both solutions can not be physical simultaneously. For example if one refers to Eq.(10) one notices (depending on the sign of β) one of Ψ_R, Ψ_L diverges as m goes to zero such that singling out one of them.

This, in our opinion, may provide clue for why the right handed component of neutrinos are very small. This result also gives an argument in favor of the unnaturalness of exactly massless neutrinos.

As we have mentioned in the introduction the framework introduced here has some conceptual similarities with the study of Dvali and Shifman [3]. They simply assume that there are more than one brane in the fifth dimension without giving an explicit model which realizes this. They take the extra dimension to be compact and they do not consider the effect of gravity. Because of experimental constraints [11] these brains must be close in the extra dimension if they all contain the standard model particles. This makes neglecting the gravity difficult and the stablization of these brains a more subtle question. Moreover they give their analysis on general grounds. Of course this approach has some advantages such as providing a general framework for future studies. However we believe that the introduction of a more specific scheme will be phenomenologically more promising. As we have mentioned in the introduction another study which takes different fermions differ by their locations in the extra dimension(s) is given by Arkani-Hamed and Schmaltz [4]. Although they do not study the problem of fermion familes and chirality their study has some common conceptual points with the present one. However they also do not consider the effect of gravity and their space is compact. Moreover the fermion masses for different fermions are simlpy taken different to put them in different locations in the extra dimension(s) while in the present model different fermion masses naturally arise as a result of the non-trivial form of the domain wall. However the models by Ref.[3] and Ref.[4] are stronger than the present model in one aspect; they obtain the fermion masses by using the technique of the overlap of wave functions as we do not introduce a method to derive the fermion masses. One can employ the same technique to obtain the fermion masses in this model. We leave this point open to facilate consideration of different options as well in future.

3 Inclusion of Gravity

After pointing out that $m\gamma_5$ contributes to fermion masses in higher (e.g. 7 or higher) dimensions we return back to five dimensions to consider the issue of including gravity into the picture. In the previous section the issue of confinement of gravity (at least at low energies) to the usual four dimensional space-time is not considered. So the next step is the modification of our metric given in Eq.(2) to a Randall-Sundrum-like form

$$\begin{aligned} ds^2 &= e^A dx_\mu \, dx^\mu - (3ay^2+b)^2 e^B \, dy^2 \\ &= e^{-2\eta(y)} dx_\mu \, dx^\mu - (3ay^2+b)^2 e^{-6\eta(y) - 2tanh\eta(y) + \frac{2}{3} tanh^3 \eta(y)} dy^2 \end{aligned} \tag{13}$$

where under the assumption of orbifold symmetry

$$\eta(y) = \frac{\mu}{\sqrt{2}} |(ay^3 + by + c)| \tag{14}$$

Unlike the Randall-Sundrum model we take only one brane, that is, the domain wall due to ϕ_{cl}. The relavant action is

$$S = \int d^5x \sqrt{-G} (\Lambda + \mathcal{L}_{cl}) \tag{15}$$

where $\mathcal{L}_{cl}$ stands for the Lagrangian of the classical fields ϕ_{cl} and σ_{cl} given in Eq.(13), G is the five dimensional metric tensor, and R is the five dimensional Ricci scalar, Λ stands for the cosmological constant in the bulk. We take ϕ_{cl}, σ_{cl} to be the classical solutions of ϕ and σ given in the previous section. This is plausible if we assume that μ in Eq.(15) is very small and the metric in Eq.(2) is an approximation of Eq.(13) for a sufficiently broad range of y values where $\eta \simeq 0$. Another view may be to consider the metrics in Eq.(2) and Eq.(15) be different the particular forms of a time dependent metric at two different times. For example one may multiply A and B in Eq.(15) by $\frac{1}{2}(1 + tanh\, x^0)$. Then the metrics Eq.(2) and Eq.(15) correspond to its value at $x^0 \to -\infty$ and $x^0 \to +\infty$, respectively. Hence one can suppose that the classical solutions ϕ_{cl} and σ_{cl} are created at $x^0 << 0$ and they survive at the present (Eq.(15)) where $x^0 >> 0$. Of course it is preferable to get the exact domain wall solution corresponding to the above metric to get a full view of the model but the corresponding equations

seem to be rather diffucult to solve and one may need to inculde additional scalar fields to the spectrum to satisfy the Einstein equations in this general case. This makes the analysis even more complicated. In fact the analysis of the model for a sufficiently wide range of y values is enough to see the essential points of the model.

After replacing ϕ_{cl} and σ_{cl} in $\mathcal{L}_{cl}$ (which is given in Eq.(13)) one notices that $\mathcal{L}_{cl}$ reduces to $\mathcal{L}_{cl}=-\frac{\mu^4}{2\lambda}[(1-tanh^2\eta)^2+\frac{1}{2})]$. The 55 component of the energy-momentum tensor is $\frac{\mu^4}{2\lambda}(\eta')^2[(1-tanh^2\eta)^2+1]+(\eta')^2\mathcal{L}_{cl}=\frac{\mu^4}{4\lambda}(\eta')^2$ where η' denotes the derivative of η with respect to y. The corresponding Einstein equations are

$$R_{MN}-\frac{1}{2}G_{MN}R=\frac{1}{4M^3}[G_{MN}(\Lambda+\mathcal{L}_{cl})-\partial_M\phi_{cl}\partial_N\phi_{cl}]-\partial_M\sigma_{cl}\partial_N\sigma_{cl}] \quad (16)$$

Under the assumption of the metric given in (13) the Einstein equations [12,13] are satisfied for all y provided

$$\Lambda=0, \quad , \quad 48\lambda M^3=\mu^2 \quad (17)$$

Another point worth to mention is that the particles are confined at the center of the domain wall due to gravity as well. In order to see this more clearly let us study the equation of motion for graviton zero modes. As in Ref.1 we write the metric tensor with linearized quantum fluctuations included as $g_{MN}=G_{MN}+h_{MN}$. h can be written as $h_{MN}=\epsilon_{MN}(y)\,e^{ip.x}$ where $p^2=m^2$. We know that $h_{\mu\nu}$ must have a massless mode corresponding to the usual gravity. So this graviton zero mode must be confined to the brane in order to prevent any conflict with inverse square law of gravity. For this purpose one must write the linearized equation of motion for $h_{\mu\nu}$. We work in the gauge $\partial^\mu h_{\mu\nu}=h^\mu_\mu=0$ as in Ref1. We expand the four dimensional metric tensor as $g_{\mu\nu}=e^{-2\eta(y)}\eta_{\mu\nu}+h_{\mu\nu}$ and $g_{55}=-(3ay^2+b)^2e^{-6\eta(y)-tanh\eta(y)+\frac{1}{3}tanh^3\eta(y)}+h_{55}$ where $\eta_{\mu\nu}$ is the Minkowski metric tensor. The equation of motion for $h_{\mu\nu}$ is

$$\begin{aligned}&[\frac{m^2}{2}e^{2\eta(y)}-\frac{1}{2(3ay^2+b)^2}e^{6\eta(y)+tanh\eta(y)-\frac{1}{3}tanh^3\eta(y)}\partial_y^2\\&-\frac{\mu^4}{2\lambda}(\eta')^2[(1-tanh^2\eta)^2+1]]\epsilon_{\mu\nu}=0\end{aligned} \quad (18)$$

This is equivalent to

$$[-\frac{1}{2}\partial_y^2 + V(y)]\epsilon_{\mu\nu} = 0 \tag{19}$$

where

$$V(y) = \frac{1}{2}m^2 e^{2\eta(y)}(3ay^2+b)^2 e^{-6\eta(y)-2tanh\eta(y)+\frac{2}{3}tanh^3\eta(y)}$$
$$-(3ay^2+b)^2 e^{-6\eta(y)-2tanh\eta(y)+\frac{2}{3}tanh^3\eta(y)}\frac{\mu^4}{2\lambda}(\eta')^2[(1-tanh^2\eta)^2+1] \tag{20}$$

One notices that the potential V is attractive for for large y's while it is repulsive for small y's. In this way one can account for why the gravitational attraction is small in our universe while the graviton zero-mode is localized in the fifth dimension. Another property of the above potential is that the massive modes are less localized and their localization peak is farther to the center of the domain wall than that of the zero mode.

The Dirac equation is

$$ie^{\eta(y)}\gamma^\mu D_\mu\Psi + \frac{1}{(3ay^2+b)}e^{3\eta(y)+tanh\eta(y)-\frac{1}{3}tanh^3\eta(y)}\gamma_5\frac{\partial\Psi}{\partial y} + g\phi_{cl}\Psi = 0 \tag{21}$$

We assume that

$$ie^{\eta(y)}\gamma^\mu D_\mu\Psi = m[\eta(y)-f]\gamma_5\Psi \tag{22}$$

The y dependence of Ψ for this metric changes because the equation (9) changes into

$$\frac{\partial\Psi_R}{\partial y} - \frac{\partial\Psi_L}{\partial y} + (\eta(y)-f)(3ay^2+b)e^{-3\eta(y)-tanh\eta(y)+\frac{1}{3}tanh^3\eta(y)}[\alpha\sigma_{cL} + g\phi_{cl}]\Psi_R$$
$$+(\eta(y)-f)(3ay^2+b)e^{-3\eta(y)-tanh\eta(y)+\frac{1}{3}tanh^3\eta(y)}[-\alpha\sigma_{cl} + g\phi_{cl}]\Psi_L = 0 \tag{23}$$

The y dependence corresponding to this equation is essentially the same, in its form, as in the previous section. We do not give the explicit y dependence of Ψ here because it is too long. The new mass term for fermions becomes

$$[\eta(y_i)-f]e^{\eta(y_i)}\ m\,\gamma_5 \quad , \quad i=1,2,3 \tag{24}$$

where y_i denotes the location of the i'th family in the fifth dimension. The masses of different families (after embeding the model in a higher dimensional space) can be made differ significantly in this case due to the additional exponent. Notice this was not the case in the previuos section because we have to take y_i's close to each other due to experimental constraints on the scale of the observable part of extra dimensions. The result that the fermion masses are larger for larger values of y may seem surprising at first sight in a Kaluza-Klein picture. On the other hand it is evident from equation (10) that the more massive fermions are much more localized in the fifth dimension. So there is no conflict with Kaluza-Klein view of fermion masses. In fact the result that very massive fermions are located much farther than the usual fermions may explain the reason behind the small mixture of massive fermions with low mass fermions.

4 Conclusion

We have seen that there is a considerable hope for explaining the origin of fermion families and chirality by using domain walls in extra dimensions. We think that one of the most important virtues of the present model is that it reaches almost all of its conclusions through explicit formula instead of a vague picture. However there is still a long way to go to put this scheme in a more detailed phenomenological model which can give realistic description of chirality and fermion families in the context of standard model. Probably in such a description one should take the gauge bosons corresponding to weak interactions to be localized on the sub-brane containing the left-handed brane while the gauge bosons of the non-chiral interactions can freely propagate over the whole (mother) 3-brane. Such an attempt may need to embed this simple scheme in higher dimensions. Such an extension may be done by giving similar constructions and solving the corresponding equations for vortices [14] or other topological defects. Another, maybe simpler, route to go is to take the topological defect in higher dimension to be a domain-wall junction or similar intersections

of multi branes in higher dimensions [15]. All these points will be clarified by further studies in future.

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